Three-Sphere Low Reynolds Number Swimmer with a Cargo Container

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A recently introduced model for an autonomous swimmer at low Reynolds number that is comprised of three spheres connected by two arms is considered when one of the spheres has a large radius. The Stokes hydrodynamic flow associated with the swimming strokes and net motion of this system can be studied analytically using the Stokes Green's function of a point force in front of a sphere of arbitrary radius R provided by Oseen. The swimming velocity is calculated, and shown to scale as $1/R^3$ with the radius of the sphere.

Since the pioneering work of G. I. Taylor [1] we know that swimming at low Reynolds number is a nontrivial task, which is also echoed in the incredibly sophisticated mechanisms involved in swimming of bacteria [2, 3]. While nature could overcome the difficulties of swimming at low Reynolds number by taking advantage of a continuum of degrees of freedom, this will be much more challenging for an artificially engineered model, as it could often use a finite number of degrees of freedom like most manmade devices. The relation between the number of degrees of freedom and swimming at low Reynolds number was considered by Purcell in his seminal work [4], who concluded that one degree of freedom is not enough and the minimum required is two. He further elaborated on this by proposing a simple swimmer model made of three joined rods, whose motion was fully analyzed only recently [5] due to the complicated nature of the hydrodynamic problem of three rotating finite rods [6].

If we take advantage of the positional degrees of freedom rather than the orientational ones, we can construct swimmer models that can be much more easily analyzed. This is the principle used in the design of a simple swimmer that is made of three spheres connected by two arms that open and close [7]. The simplicity of the three-sphere swimmer model allows us to study various characteristic properties of the swimmer itself—such as the effects of external load and noise, stress distribution, power consumption, etc.—in great details [8, 9]. Moreover, it also makes it possible to study the interaction between two [10] or more of these swimmers, which could be used towards a systematic study of the floc behavior of a suspension of such swimmers [11, 12, 13, 14]. A similar principle has been used to propose other swimmer models that can also be easily analyzed [15, 16, 17]. This class of swimmer models uses a finite number of compact degrees of freedom, and is to be contrasted from those that use a continuum [18, 19]. An artificial microscopic swimmer has been recently made using magnetic beads connected by DNA linkers, and has been shown to be able to propel itself via beating strokes that are induced by an oscillating magnetic field [20]. It is also possible to design swimmers with no moving parts by taking advantage of phoretic phenomena, as has been proposed and studied theoretically [21, 22, 23] and also realized experimentally [24, 25, 26].

The analysis of the three-sphere swimmer model has so far been made in the limit where the radii of the spheres are the smallest length scales in the system, namely, they are much smaller than the distances between the spheres. In practical applications, however, one can imagine that the swimmer could be used to carry a cargo, which would be possible if one of the spheres was blown into a large hollow spherical shell, with a radius that is comparable to the distance between the spheres. A similar geometry could also be useful for experimental testing of the swimmer design, as a large spherical bead as a part of the swimmer would allow optical probing of its propelled motion. With these motivations, here we study the swimming of the three-sphere model with one of the spheres having a finite radius comparable to the other length scales. It is possible to fully analyze the motion of this model swimmer along the same lines as the simpler version, thanks to the Stokes Green's function of a point force in front of a sphere of arbitrary radius calculated by Oseen in 1927 using the method of images [27]. We find that the average swimming velocity scales as $1/R^3$ with the radius of the container. Our results show that a simplistic view of "estimating" the velocity of an autonomous swimmer by dividing the propelling force in the system by its Stokes friction coefficient is incorrect, and a only proper analysis of the hydrodynamic flow due to the deformations of the swimmer should be used to determine the propulsion velocity.

Consider the geometry of Fig. 1 where one of the spheres in the swimmer is in the form of a large container of radius R and the other two have radii a_1 and a_2. To calculate the swimming velocity of the machine, we work in the frame of reference that is co-moving with the large sphere. Assuming that there are two force centers located at $\mathbf{x}_1$ and $\mathbf{x}_2$ exerting the forces $\mathbf{f}_1$ and $\mathbf{f}_2$, respectively, we can write the velocity of the fluid at any

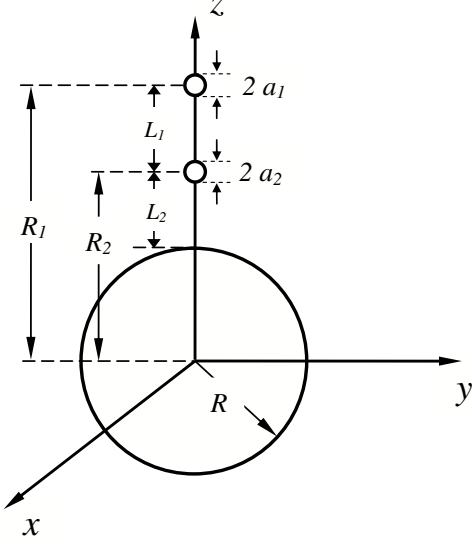*Electronic address: r.golestanian@sheffield.ac.uk

FIG. 1: Schematics of the swimmer with a cargo container. The instantaneous distance between the two small spheres and the center of the large sphere are $R_1(t)$ and $R_2(t)$, and there three spheres are assumed to always be positioned along the z-axis.

point $\mathbf{x}$ as

$$
\begin{aligned}
v_i(\mathbf{x}) = {} & -V_i + \frac{1}{4}\frac{R}{r}\left(3 + \frac{R^2}{r^2}\right)V_i \\
& + \frac{3}{4}\frac{R}{r}\left(1 - \frac{R^2}{r^2}\right)\frac{x_i x_j}{r^2}V_j \\
& + \frac{1}{8\pi\eta}\left[G_{ij}(\mathbf{x}, \mathbf{x}_1)f_{1j} + G_{ij}(\mathbf{x}, \mathbf{x}_2)f_{2j}\right], \quad (1)
\end{aligned}
$$

where $\mathbf{V}$ is the velocity of the large sphere in the laboratory frame, and $G_{ij}(\mathbf{x}, \mathbf{x}')$ is the Oseen Green's function that vanishes on the surface of the large sphere [27, 28].

Assuming that the two small spheres are located on the z-axis (see Fig. 1) at $\mathbf{x}_1 = (0, 0, R_1)$ and $\mathbf{x}_2 = (0, 0, R_2)$, symmetry requires that the device swims along the z-direction, namely, $\mathbf{V} = (0, 0, V)$. Using no-slip boundary condition on the two small spheres, and defining the quantities

$$
C_\alpha = 1 - \frac{3}{2}\frac{R}{R_\alpha} + \frac{1}{2}\frac{R^3}{R_\alpha^3}, \quad (2)
$$

we can find their velocities as

$$
v_\alpha = -C_\alpha V + \frac{1}{8\pi\eta}\mathcal{H}_{\alpha\beta}f_\beta, \quad (3)
$$

where $\mathcal{H}_{\alpha\beta} = G_{zz}(\mathbf{x}_\alpha, \mathbf{x}_\beta)$ for $\alpha, \beta = 1, 2$, and summation over repeated indices is understood. Note that $\mathcal{H}_{12} = \mathcal{H}_{21}$ due to the symmetry of the Green's function. Assuming that the small spheres have prescribed velocities $v_\alpha = \dot{R}_\alpha$, we can determine the forces as

$$
f_\alpha = 8\pi\eta\,\mathcal{H}_{\alpha\beta}^{-1}\left(\dot{R}_\beta + C_\beta V\right), \quad (4)
$$

where $\mathcal{H}_{\alpha\beta}^{-1}$ represent the components of the inverse matrix for $\mathcal{H}_{\alpha\beta}$, with respect to the α, β indices. Note that strictly speaking Faxen's theorem [29] requires that an additional contribution proportional to the Laplacian of the velocity field is added to the expression for the velocity of the small spheres. However, these contributions will generate terms that are higher order in the ratio between the sizes of the small spheres and the distances in the system, and can hence be neglected here.

We can now impose the condition of zero external force or the vanishing of the total Stokeslet strength as [30]

$$
C_\alpha f_\alpha + 6\pi\eta RV = 0, \quad (5)
$$

which yields the final equation. The above equation can be understood as the vanishing of the sum of the forces exerted on the fluid at the outer surface of the sphere. When we have extended objects rather than just point forces, we should enforce the force-free condition by looking at the stress field in the vicinity of the swimmer and show that its integral over any closed surface vanishes. In our case here, this stress will have a contribution from the sphere, which yields the familiar Stokes force term in Eq. (5), plus contributions that have propagated from the point forces that are located at distances R_1 and R_2. This form of the force-free constraint can be related to the case of three point-like spheres, where the force-free relation reads $f_1 + f_2 + f_3 = 0$. To this end, we can simply take the limit of $R \to 0$ in Eq. (5) and note that the viscous force term will produce f_3 in this limit as V is the velocity of the third sphere. Equation (5) proves that it is incorrect to assume that the velocity of the bead can be extracted from a force balance between the contributions from the two force centers and the friction force at the bead, which would have (falsely) implied the vanishing of $\sum_\alpha f_\alpha + 6\pi\eta RV$.

Putting in the forces from Eq. (4), Eq. (5) can be solved to yield to propulsion velocity as

$$
V = -\frac{C_\alpha \mathcal{H}_{\alpha\beta}^{-1}\dot{R}_\beta}{C_\alpha \mathcal{H}_{\alpha\beta}^{-1}C_\beta + \frac{3}{4}R}. \quad (6)
$$

We also feed this result back into Eq. (4) to find the forces at the two small spheres. We find

$$
f_\alpha = 8\pi\eta\left[\mathcal{H}_{\alpha\beta}^{-1}\dot{R}_\beta - \frac{\mathcal{H}_{\alpha\beta}^{-1}C_\beta C_\gamma \mathcal{H}_{\gamma\delta}^{-1}\dot{R}_\delta}{C_\gamma \mathcal{H}_{\gamma\delta}^{-1}C_\delta + \frac{3}{4}R}\right], \quad (7)
$$

where $\gamma, \delta = 1, 2$.

To calculate the swimming velocity we need the explicit expression for $G_{zz}(z, z')$, which is presented in the 1927 monograph by Oseen [27]. It reads

$$
\begin{aligned}
G_{zz}(z, z') = {} & \frac{2}{|z - z'|} - \frac{2R}{|zz' - R^2|} \\
& - \frac{R(z^2 - R^2)(z'^2 - R^2)}{(zz' - R^2)^3}, \quad (8)
\end{aligned}
$$

which yields the following expressions for the components of the $\mathcal{H}$-matrix

$$\mathcal{H}_{11} = \frac{4}{3a_1} - \frac{3R}{(R_1^2 - R^2)}, \tag{9}$$

$$\mathcal{H}_{22} = \frac{4}{3a_2} - \frac{3R}{(R_2^2 - R^2)}, \tag{10}$$

$$\mathcal{H}_{12} = \frac{2}{|R_1 - R_2|} - \frac{2R}{(R_1 R_2 - R^2)} - \frac{R(R_1^2 - R^2)(R_2^2 - R^2)}{(R_1 R_2 - R^2)^3}, \tag{11}$$

where the terms involving a_1 and a_2 appear with a different prefactor according to standard treatments of the Oseen tensor to extract the friction coefficient. Putting the explicit forms of the $\mathcal{H}_{\alpha\beta}$ matrix elements as functions of the instantaneous separations $R_1(t)$ and $R_2(t)$ in Eq. (6), we find a closed form expression for the instantaneous swimming velocity.

It is instructive to study the asymptotic limits of Eq. (6) in various limits. In the limit where $(a_1, a_2, R) \ll (R_1, R_2, R_1 - R_2)$, we can use $R_2 = \ell_2 + u_2$ and $R_1 = \ell_1 + \ell_2 + u_1 + u_2$ and expand the expression in powers of u_α. Keeping only terms up to the leading order, we find the average velocity as

$$\overline{V} = -\frac{3}{2}\frac{a_1 a_2 R}{(a_1 + a_2 + R)^2}\left[\frac{1}{\ell_1^2} + \frac{1}{\ell_2^2} - \frac{1}{(\ell_1 + \ell_2)^2}\right] \times \overline{(u_1 \dot{u}_2 - \dot{u}_1 u_2)}, \tag{12}$$

where the averaging over a complete swimming cycle with period T is defined by $\overline{V} = \frac{1}{T}\int_0^T dt V(t)$. This is identical to the result obtained previously [8] using the standard form of the Oseen tensor, which is a good check for our result. (The apparent sign difference with the previous result [8] is due to a different convention for the direction of the z-axis.) Note that in this case the swimming velocity scales as the inverse of the second power of the largest length scale in the system, namely $1/\ell^2$.

In the limit where $(a_1, a_2) \ll (R_1 - R, R_2 - R) \ll R$, we can use $R_1 = R + L_1 + L_2$ and $R_2 = R + L_2$ (see Fig. 1), and expand the expression in powers of L_α/R. We find the instantaneous velocity as

$$V = \frac{9}{2}\left(\frac{a_1 a_2}{R^3}\right)\frac{L_2^2(3L_1 + 4L_2)}{L_1(L_1 + 2L_2)^3} \times \left[(\dot{L}_1 + \dot{L}_2)L_2^2 + \dot{L}_2(L_1 + L_2)^2\right], \tag{13}$$

to the leading order. Expanding for small deformations using $L_1 = \ell_1 + u_1$ and $L_2 = \ell_2 + u_2$, yields

$$\overline{V} = \frac{9}{4}\left(\frac{a_1 a_2}{R^3}\right)\frac{\ell_2^2(3\ell_1 + 2\ell_2)}{\ell_1^2(\ell_1 + 2\ell_2)}\overline{(\dot{u}_1 u_2 - \dot{u}_2 u_1)}. \tag{14}$$

Equations (13) and (14) are proportional to $1/R^3$, which is in marked contrast to the $1/\ell^2$ scaling in the case of a (more) symmetric swimmer. The difference in the scaling suggests that it might be more efficient to distribute the size of the swimmer across the three spheres, rather than concentrate all of it on one sphere. Such a statement, however, needs to be checked in the limit of a symmetric swimmer with spheres of large radii compared to their surface-to-surface separations. Equation (14) also shows that it is beneficial for the two small spheres to be close to each other and relatively farther away from the surface of the sphere, i.e. $\ell_2 \gg \ell_1$. Note, however, that this result [Eq. (14)] is only valid for $(a_1, a_2) \ll (\ell_1, \ell_2) \ll R$, which means that the previous prescription for creating optimal swimming efficiency should be written as $(a_1, a_2) \ll \ell_1 \ll \ell_2 \ll R$ rigorously speaking.

The present calculation can be readily generalized to the case of one large sphere with many small spheres. For a line-up of all the spheres along the z-axis, Eqs. (2), (3), (4), (5), (6), and (7) will hold true provided the indices are run over the number of small spheres in the system. For more complicated assortments of small spheres, one can still use this formalism provided the right tensorial structure is maintained along the way. A useful particular case could be a line-up of small spheres that oscillate laterally, i.e. perpendicular to the z-axis, which would mimic the beating motion of a (finite) tail. It will also be interesting to consider the possibility of a number of such structures attached to the bead interacting with each other hydrodynamically. These "active tails" could potentially synchronize with each other under certain conditions, to generate an efficient propulsion. Our systematic approach might be able to provide useful insight into the intriguing phenomenon of collective hydrodynamic patterns and phases [31, 32, 33]. One can also study the efficiency of the swimmer in terms of power consumption and useful work, which for this swimmer follows very closely similar studies that are presented elsewhere [8].

In conclusion, we have generalized the simple model of a low Reynolds number swimmer to the case where one of the spheres is large, and analyzed its motion using the Oseen tensor of a point force in front of a sphere which imposes no-slip boundary condition on the Stokes flow. The results found here could help us in possible experimental probes of self-propelled devices.

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